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Bethe ansatz for higher-spin *XYZ* models—low-lying excitations

Takashi Takebe†

Department of Mathematical Sciences, University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo, 153, Japan

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Abstract. A higher-spin generalization of the *XYZ* spin chain defined by the fusion procedure is considered. The energy and momentum of the ground state and low-lying excited states of the model are computed by means of an isomorphism between a space of theta functions on which the Sklyanin algebra acts and a space obtained by the fusion procedure.

1. Introduction

In this paper we calculate energy and momentum of low-lying excited states of the higher-spin generalization of the XYZ spin chain model by means of the algebraic Bethe ansatz method. This work is a continuation of the author's previous works [1, 2], where a generalization of the eight-vertex model [3] was studied and the two-particle S-matrix of the corresponding spin chain model was computed. There we assigned a $(2l+1)$-dimensional space to the vertical edges of the lattice and a two-dimensional space to the horizontal edges. In the present paper we make use of the fusion procedure (see [4–7]) to assign a higher-dimensional space to the horizontal edges and then consider the corresponding one-dimensional quantum spin chain model.

In section 2 we shall give an explicit isomorphism of a representation space of the Sklyanin algebra which is defined as a space of theta functions with the space of symmetric tensors. (This is a special case of more general isomorphisms by Hasegawa [8].) Through this identification we can identify the L operator of the model in [2] with a special case of R matrices in [6], and use the results in [2] to compute the general models. In particular this identification is indispensable for the interpretation of a special value of the logarithm of a transfer matrix and its logarithmic derivative as a momentum and an energy operator. See, e.g., [9].

As is conjectured in [2] from the corresponding results for the higher-spin XXX and XXZ models [10–12], the energy and the momentum are independent of the spin of the local quantum space and are expressed as the sum of two terms, each of which depends on the rapidity of a hole in the string configuration. Thus we can justify the interpretation of those states as 'two-particle states'.

We make use of the results in [2], but adopt simpler normalizations in [13]. In particular, we normalize the R matrices by the unitarity condition, and therefore they are meromorphic functions of the spectral parameter, while [6] and [12] use holomorphic R matrices.

† Present address: Department of Mathematics, University of California, Berkeley, CA94720, USA (until August 1997).

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2. Fusion procedure

In this section, we briefly review the fusion procedure for the elliptic R matrices.

Let $V^l = \mathrm{Sym}(V_1 \otimes \cdots \otimes V_{2l})$ for $l \in \frac{1}{2}\mathbb{Z}$, where $V_i \cong \mathbb{C}^2$ $(i = 1, \ldots, 2l)$ and Sym is the symmetrizer. The elliptic R matrices $R^{l,l'}(u)$ have the following properties.

(i) $R^{l,l'}(u)$ is a linear endomorphism of $V^l \otimes V^{l'}$ meromorphically depending on a complex parameter u.

(ii) (*Yang–Baxter equation*) As an endomorphism of $V^l \otimes V^{l'} \otimes V^{l''}$

$$
\begin{aligned}
&R^{l,l'}_{12}(u_1 - u_2) R^{l,l''}_{13}(u_1 - u_3) R^{l',l''}_{23}(u_2 - u_3) \\
&\qquad = R^{l',l''}_{23}(u_2 - u_3) R^{l,l''}_{13}(u_1 - u_3) R^{l,l'}_{12}(u_1 - u_2).
\end{aligned}
\tag{2.1}
$$

(iii) (*Unitarity*) As an endomorphism of $V^l \otimes V^{l'}$

$$
R^{l,l'}_{12}(u - v) R^{l',l}_{21}(v - u) = \mathrm{Id}_{V^l \otimes V^{l'}}\,. \tag{2.2}
$$

(iv) When $u = 0$, $R^{l,l}(u)$ is a permutation operator: for all $v, w \in V^l$

$$
R^{l,l}(0)(v \otimes w) = w \otimes v. \tag{2.3}
$$

They are constructed in [5–7] by the fusion procedure [4] from Baxter's R matrix $R^{1/2,1/2}(u) = R(u;\tau)$ defined by

$$
R(u) = \sum_{a=0}^{3} W_a(u) \sigma^a \otimes \sigma^a \qquad W_a(u) := \frac{\theta_{g_a}(u;\tau)\theta_{11}(2\eta;\tau)}{2\theta_{g_a}(\eta;\tau)\theta_{11}(u+2\eta;\tau)} \tag{2.4}
$$

where $g_0 = (11)$, $g_1 = (10)$, $g_2 = (00)$, $g_3 = (01)$. The explicit definition of $R^{l,l'}$ is

$$
\begin{aligned}
R^{1/2,l'}_{V_i,V^{l'}}(u) &:= \mathrm{Sym}_{\overline{2l'}\ldots\bar{1}}\, R_{V_i,V_{\overline{2l'}}}(u + (2l'-1)\eta) \cdots \\
&\qquad \cdots R_{V_i,V_{\bar{j}}}(u + (2j - 2l' - 1)\eta) \cdots R_{V_i,V_{\bar{1}}}(u + (-2l'+1)\eta)
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
R^{l,l'}_{V^l,V^{l'}}(u) &:= \mathrm{Sym}_{1\ldots 2l}\, R^{1/2,l'}_{V_{2l},V^{l'}}(u + (2l-1)\eta) \cdots \\
&\qquad \cdots R^{1/2,l'}_{V_j,V^{l'}}(u + (2j - 2l - 1)\eta) \cdots R^{1/2,l'}_{V_1,V^{l'}}(u + (-2l+1)\eta).
\end{aligned}
\tag{2.6}
$$

Here $V_i \cong V_{\bar{j}} \cong \mathbb{C}^2$, the suffices of R designate the spaces on which the R matrix acts and $\mathrm{Sym}_{1\ldots m}$ is the symmetrizer on the space $V_1 \otimes \cdots \otimes V_m$, etc.

There is another expression of $R^{1/2,l'}(u)$ in terms of the representation of the Sklyanin algebra which we used in [2]. The Sklyanin algebra [14] $U_{\tau,\eta}(sl(2))$ is generated by four generators S^0, S^1, S^2 and S^3 satisfying the relations coming from the Yang–Baxter-type relation of the L operator, $L(u)$, defined by

$$
L(u) = \sum_{a=0}^{3} W^L_a(u) \sigma^a \otimes S^a \qquad W^L_a(u) = \frac{\theta_{g_a}(u)}{2\theta_{11}(2\eta)\theta_{g_a}(\eta)}. \tag{2.7}
$$

The Sklyanin algebra has a representation on a space of theta functions

$$
\rho^{(l)} : U_{\tau,\eta}(sl(2)) \to \mathrm{End}_{\mathbb{C}}(\Theta^{4l+}_{00}) \tag{2.8}
$$

$$
\Theta^{4l+}_{00} = \left\{ f(y)\text{: holomorphic on } \mathbb{C} \;\middle|\; \begin{array}{l} f(y+1) = f(-y) = f(y) \\ f(y+\tau) = \mathrm{e}^{-4l\pi\mathrm{i}(2y+\tau)} f(y) \end{array} \right\}. \tag{2.9}
$$

It is easy to see that $\dim \Theta_{00}^{4l+} = 2l+1$. The generators S^a act on this space as difference operators. See [15] and appendix A of [2], [13]. We fix a basis of Θ_{00}^{2+}, $(\theta_{00}(2y;2\tau) - \theta_{10}(2y;2\tau), \theta_{00}(2y;2\tau) + \theta 10(2y;2\tau))$ and identify Θ_{00}^{2+} with $\mathbb{C}^2$ through this basis. Then we fix an isomorphism of the space of symmetric tensors and the spin l representation space of the Sklyanin algebra as follows:

$$
\begin{aligned}
V^l &= \mathrm{Sym}(V_1 \otimes \cdots \otimes V_{2l}) \ni \mathrm{Sym}(f_1(y_1) \otimes \cdots \otimes f_{2l}(y_{2l})) \\
&\mapsto f_1(y) \cdots f_{2l}(y) \in V^l = \Theta_{00}^{4l+}
\end{aligned}
\tag{2.10}
$$

where $V_i \cong \mathbb{C}^2$ $(i = 1, \ldots, 2l)$ identified with Θ_{00}^{2+}, $f_i(y_i) \in V_i$ and Sym is the symmetrizer. Under this identification the L operator (2.7) is proportional to the R matrix defined by (2.5):

$$
R^{1/2,l}(u) = \frac{\theta_{11}(2\eta)}{\theta_{11}(u + (2l+1)\eta)} \, \mathrm{Id}_{\mathbb{C}^2} \otimes \rho^{(l)}(L(u+\eta)). \tag{2.11}
$$

This can be verified by comparing the action of both sides on the intertwining vectors. (See lemma 2.1.3 and theorem 2.3.3 of the third reference in [6] for the left-hand side and equations (1.18)–(1.21) of [13] for the right-hand side. See also [8] for the general cases.)

As is the case with the trigonometric and rational R matrix, there is a recurrence relation with respect to the auxiliary spin [16, 12]. Let $R^{l,l'}(u)$ and $R^{1/2,l'}(u)$ be the R matrices on the spaces $\mathrm{Sym}(V_{2l} \otimes \cdots V_1) \otimes V^{l'}$ and $V_0 \otimes V^{l'}$, respectively. Here $V_i \cong \mathbb{C}^2$ $(i = 1, \ldots, 2l)$ and $V^{l'}$ is a space of symmetric tensors defined above. Then as an operator on $\mathrm{Sym}(V_{2l} \otimes \cdots V_1) \otimes V_0 \otimes V^{l'}$

$$
\begin{aligned}
&R^{l,l'}(u+\eta) R^{1/2,l'}(u - 2l\eta) \\
&\quad = \left(\begin{array}{c|c} R^{l+1/2,l'}(u) & 0 \\ \hline * & \begin{array}{c} \text{q-det}\, R^{1/2,l'}(u - (2l-1)\eta) \\ \times R^{l-1/2,l'}(u+2\eta) \end{array} \end{array} \right)
\end{aligned}
\tag{2.12}
$$

where q-det R is the quantum determinant [17] defined by

$$
\text{q-det}\, R^{1/2,l'}(u) = \mathrm{tr}_{01}\, P_{01}^- R^{1/2,l'}(u+\eta) R^{1/2,l'}(u-\eta) = \frac{\theta_{11}(u - 2l'\eta)}{\theta_{11}(u + 2l'\eta)} \, \mathrm{Id}_{V^{l'}} \tag{2.13}
$$

(P_{01}^- is a projection to the antisymmetric tensor in $V_1 \otimes V_0$). The block structure of the right-hand side of (2.12) comes from the decomposition of the tensor product of the auxiliary spaces by the Young symmetrizers as follows:

$$
\boxed{1 \,|\, 2 \,|\, \cdots \,|\, 2l} \otimes \boxed{0} = \boxed{0 \,|\, 1 \,|\, 2 \,|\, \cdots \,|\, 2l} \oplus \begin{array}{l} \boxed{1 \,|\, 2 \,|\, \cdots \,|\, 2l} \\ \boxed{0} \end{array} . \tag{2.14}
$$

Here we denote the image of the Young symmetrizer by the corresponding Young tableau. Since the second Young symmetrizer on the right-hand side of (2.14) gives an isomorphism from the space $\mathrm{Sym}(V_{2l} \otimes \cdots \otimes V_2) \otimes \mathrm{Ant}(V_1 \otimes V_0) \cong \mathrm{Sym}(V_{2l} \otimes \cdots \otimes V_2)$ to its image, we identify these spaces in the right lower corner of the right-hand side of (2.12). The proof of this equation is based on lemma 5 of [5] and is done by the same argument as that given in section 1 of [16].

3. Higher-spin XYZ model

In this section we define a higher-spin generalization of the XYZ model and apply the algebraic Bethe ansatz.

The state space of our model is

$$\mathcal{H} = V_1^l \otimes V_2^l \otimes \cdots \otimes V_N^l \tag{3.1}$$

where $V_j^l \cong V^l$. We define the model by specifying the transfer matrix, namely the generating function of the quantum integrals of motion, as follows:

$$T^{l',l}(u) := \mathrm{tr}_{V^{l'}} R^{l',l}_{V^{l'},V_N^l}(u) \cdots R^{l',l}_{V^{l'},V_1^l}(u) \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}) \tag{3.2}$$

where $R_{V^{l'},V_j^l}$ acts non-trivially only on the component $V^{l'} \otimes V_j^l$ of $V^{l'} \otimes \mathcal{H}$. The most important property of the transfer matrix is the commutativity relation

$$[T^{l',l}(u), T^{l'',l}(v)] = 0 \tag{3.3}$$

which is a consequence of the Yang–Baxter equation (2.1).

Thanks to equation (2.3), we can define a momentum operator p and a Hamiltonian H of the spin chain by

$$p = \frac{1}{\mathrm{i}} \log T^{l,l}(0) \qquad H = \text{constant} \times \left.\frac{\mathrm{d}}{\mathrm{d}u} \log T^{l,l}(u)\right|_{u=0} \tag{3.4}$$

where we do not fix the constant here (see [9, 18, 10–12].)

Due to the recurrence relation of the R matrix (2.12), the transfer matrix satisfies the following recurrence relation:

$$T^{l'+1/2,l}(u) = T^{l',l}(u+\eta)T^{1/2,l}(u-2l'\eta) - \frac{\theta_{11}(u+(-2l'+1-2l)\eta)}{\theta_{11}(u+(-2l'+1+2l)\eta)} T^{l'-1/2,l}(u+2\eta). \tag{3.5}$$

Hence the diagonalization problem of $T^{l',l}(u)$ reduces to that of $T^{1/2,l}(u)$.

Hereafter we assume that the elliptic modulus τ is a pure imaginary number $\tau = \mathrm{i}/t$, $t > 0$ and the anisotropy parameter $\eta = r'/r$ is a rational number. We also assume that $M = Nl$ is an integer.

It was shown in [1] that there exist vectors $\Psi_\nu(w_1, \ldots, w_M)$ depending on an integer ν and complex parameters $(w_1, \ldots, w_M)$ such that

$$T^{1/2,l}(u)\Psi_\nu(w_1, \ldots, w_M) = t^{1/2,l}(u; \nu, w_1, \ldots, w_M)\Psi_\nu(w_1, \ldots, w_M) \tag{3.6}$$

provided that $(\nu; w_1, \ldots, w_M)$ satisfy the Bethe equations:

$$\left(\frac{\theta_{11}(w_j + 2l\eta; \tau)}{\theta_{11}(w_j - 2l\eta; \tau)}\right)^N = \mathrm{e}^{-4\pi\mathrm{i}\nu\eta} \prod_{\substack{k=1 \\ k\neq j}}^{M} \frac{\theta_{11}(w_j - w_k + 2\eta; \tau)}{\theta_{11}(w_j - w_k - 2\eta; \tau)} \tag{3.7}$$

for all $j = 1, \ldots, M$. The eigenvalue $t^{1/2,l}(u)$ is equal to

$$t^{1/2,l}(u; \nu, w_1, \ldots, w_M) = \frac{Q(u-2\eta)}{Q(u)} + h(u)\frac{Q(u+2\eta)}{Q(u)} \tag{3.8}$$

where

$$Q(u) := \mathrm{e}^{-\pi\mathrm{i}\nu u} \prod_{j=1}^{M} \theta_{11}(u - w_j + \eta) \qquad h(u) := \left(\frac{\theta_{11}(u+(-2l+1)\eta)}{\theta_{11}(u+(2l+1)\eta)}\right)^N. \tag{3.9}$$

Inductively using the recurrence relation (3.5), we can prove that the eigenvalue of $T^{l',l}(u)$ is

$$t^{l',l}(u; \nu, w_1, \ldots, w_M) = \sum_{j=0}^{2l'} a_j^{l',l}(u) \frac{Q(u+(2l'+1)\eta)Q(u-(2l'+1)\eta)}{Q(u+(2l'+1-2j)\eta)Q(u+(2l'-1-2j)\eta)} \tag{3.10}$$

where $a_0^{l',l}(u) = 1$, $a_j^{l',l}(u) = \prod_{k=1}^{j} h(u+(2l'+1-2k)\eta)$.

4. The thermodynamic limit

In this section, making use of the results of [2], we compute several thermodynamic quantities of our spin chains.

First let us recall several facts concerning the solutions of the Bethe equations (3.7) that we found in [2]. These solutions satisfy the string hypothesis, which goes back to Bethe [19]. For later convenience we rescale the parameters as follows: $x = \mathrm{i}tu$, $x_j = \mathrm{i}tw_j$. In this notation the string hypothesis says that for sufficiently large N, the solutions of (3.7) cluster into groups known as A-strings, $A = 1, 2, \ldots$, with parity $\pm$ and centre $x_j^{A,\pm}$:

$$x_{j,\alpha}^{A,\pm} = x_j^{A,\pm} + 2\mathrm{i}\eta t\alpha + \mathrm{O}(\mathrm{e}^{-\delta N}) \qquad \alpha = \frac{-A+1}{2}, \frac{-A+3}{2}, \ldots, \frac{A-1}{2} \tag{4.1}$$

where $\operatorname{Im} x_j^{A,+} = 0$ and $\operatorname{Im} x_j^{A,-} = t/2$. We denote the number of A-strings with parity $\pm$ by $\sharp(A, \pm)$.

We consider the following string configurations, which are consistent with the constraints found in [20] when $\eta = r'/r$, r, r' are integers mutually coprime, r is even, r' is odd, and $2(2l+1)\eta < 1$, as assumed in [2].

- Ground state: $\nu = 0$, $\sharp(2l, +) = N/2$, $\sharp(A, \pm) = \sharp(2l, -) = 0$ for $A \neq 2l$ and centres of $2l$-strings distributed symmetrically around 0.
- Excited state I: $\sharp(2l, +) = N/2 - 2$, $\sharp(2l-1, +) = 1$, $\sharp(2l+1, +) = 1$.

  There are two holes in the distribution of $2l$-strings which are denoted by x_1 and x_2 and are regarded as continuous parameters of the configuration. The Bethe equations determine the coordinates of the centres of the $(2l \pm 1)$-string which are denoted by $x_\pm$. There are two possibilities: $(x_-, x_+) = ((x_1 + x_2)/2, (x_1 + x_2)/2)$ or $(x_-, x_+) = ((x_1 + x_2)/2, (x_1 + x_2 + 1)/2)$.
- Excited state II: $\sharp(2l, +) = N/2 - 1$, $\sharp(2l-1, +) = 1$, $\sharp(1, -) = 1$.

  There are again two holes in the distribution of $2l$-strings which are denoted by x_1 and x_2. The Bethe equations determine the coordinates of the centres of the $(2l-1)$-string and the 1-string with parity $-$ which are denoted by x_- and x_0 respectively: $(x_-, x_0) = ((x_1 + x_2)/2, (x_1 + x_2)/2)$ or $(x_-, x_0) = ((x_1 + x_2)/2, (x_1 + x_2 + 1)/2)$.

Using the results of [2], we obtain the following asymptotics of the eigenvalue of the transfer matrix $t^{l,l}(u)$ for large N. The largest eigenvalue of $t^{l,l}$ which corresponds to the ground state is

$$\frac{1}{\mathrm{i}} \log t^{l,l}(u;\ \text{ground state}) \sim N\left(\pi l + 2\pi l x(1 - 4l\eta) + 2\sum_{n=1}^{\infty} \frac{\sinh 4\pi n l\eta t \sinh \pi n t(1 - 4l\eta)}{n \sinh \pi n t \sinh 4\pi n\eta t} \sin 2\pi n x\right). \tag{4.2}$$

The excited states I and II have the same eigenvalue of the transfer matrix $t^{l,l}(u)$, namely

$$\frac{1}{\mathrm{i}} \log t^{l,l}(u;\ \text{excited state I or II}) - \frac{1}{\mathrm{i}} \log t^{l,l}(u;\ \text{ground state}) = \log\tau(x - x_1) + \log\tau(x - x_2) \tag{4.3}$$

where

$$\log\tau(x) := -\frac{\pi}{2} - \pi x - \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{n \cosh 2\pi n\eta t}. \tag{4.4}$$

(see [21] and [2] for details of the computations). By definition (3.4), the momentum and the energy of these excited states are expressed as $p = p_1 + p_2$, $H = H_1 + H_2$ where

$$p_i := -\frac{\pi}{2} + \pi x_i + \sum_{n=1}^{\infty} \frac{\sin 2\pi n x_i}{n \cosh 2\pi n \eta t} \tag{4.5}$$

$$H_i := \text{constant} \times \left(-\pi - 2\pi \sum_{n=1}^{\infty} \frac{\cos 2\pi n x_i}{\cosh 2\pi n \eta t} \right). \tag{4.6}$$

Thus they are regarded as two-particle, spin-wave modes, each particle of which has rapidity x_i, momentum p_i and energy H_i. In particular, equations (4.5) and (4.6) show that the dispersion relation of these particles do not depend on l.

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